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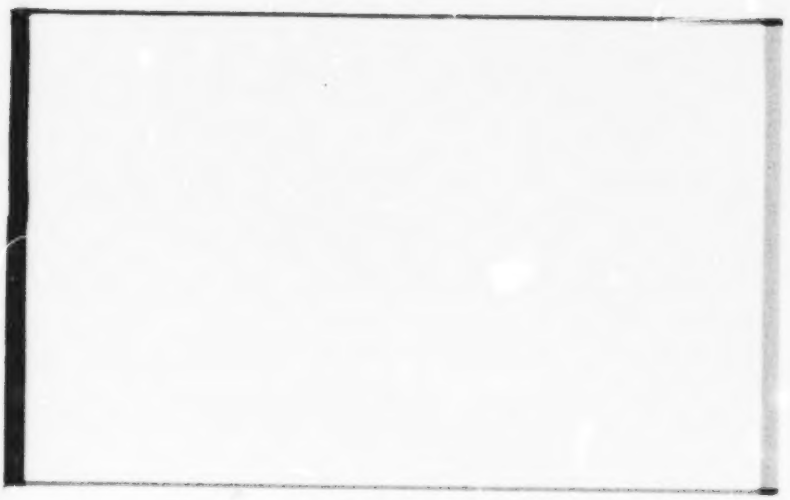

A DISCRETE-CONVEX PROGRAMMING APPROACH TO THE SIMULTANEOUS OPTIMIZATION OF LAND USE AND TRANSPORTATION

by

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ABSTRACT

Three design problems are discussed in this article. First, it is shown that the network design problem with congestion reduces to an all-or nothing traffic assignment problem under some assumptions on the congestion function and the investment cost function. Second,the land use design problem is formulated as an extension of the Koopmans-Beckmann problem and a heuristic is proposed to solve this problem. Third, it is shown that the seemingly more complex problem of designing jointly a land-use plan and a transportation network reduces to a pure land-use design problem. All that is needed to solve the joint optimization problem is a shortest path algorithm and a heuristic to solve the land use design problem. Computational experience is reported for each algorithm.

I INTRODUCTION

Much attention has been devoted in the past ten years to the problem of designing an optimal transportation network. However it has long been recognized by urban and transportation planners alike that network design should be embedded in the whole process of land use and transportation planning[1]. The performance of a land use plan depends on the transportation network and, reciprocally, the performance of the transportation network depends on the land-use plan. Therefore, the real world problem one wishes to solve is the joint optimization of a land-use plan and of a transportation plan. The first published statement of a model for this problem is due to Lundqvist (1973). The present author proposed a purely combinatorial programming approach in which both the land-use decisions and the network decisions are treated as discrete variables (see, for example, Los, 1978). For a given network, disregarding congestion, the land-use design problem was conceived as a quadratic assignment problem of the Koopmans-Beckmann type (Los, 1978). For a given land use plan the problem was conceived as an optimal network problem without congestion (Los, 1976b). The model proposed could therefore be considered as a synthesis of these two problems. If the Koopmans-Beckmann problem is chosen to model land-use plan design however, one implicitly assumes that the demand for transportation between pairs of activities is fixed. As emphasized by Hopkins (1977), this is a strong limitation of the Koopmans-Beckmann problem for the modelling of land-use decisions: in reality the demand for travel between two activities is variable. In particular it depends on transportation costs. In addition, one cannot disregard

[1] Cf. Boyce *et al.* (1970) and Harris (1967).

congestion in planning an automobile network for a city. In the model for land-use and transportation planning presented in this article, these two restrictive assumptions (no variable demand for travel and no congestion) are relaxed. The present model can be considered as a synthesis of two design models, one for land use planning and one for network design. The two separate models are also presented in this article.

The land-use decisions are treated as discrete variables, while the network variables are treated as continuous variables[1]. The resulting problem is a discrete-convex programming problem (Cf. Geoffrion, 1975). Problems of this class are in general hard to solve, but it will be shown in this article that, by projection onto the space of the discrete variables, the land-use-transportation problem reduces to a problem that is mathematically equivalent to the "pure" land-use design problem.

The rest of the paper is organized as follows. Section II presents the network design problem and shows that it can be solved by an all-or-nothing traffic assignment algorithm. Computational experience is reported. Section III presents the land-use design problem, proposes a solution method and reports preliminary computational experience. Section IV presents the land-use-transportation design model which is a natural extension of the models of sections II and III. A solution method is proposed. In section V the assumptions and limitations of the joint model are discussed. Section VI concludes this article.

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[1] This is a major difference between the model of Los (1978) and the model proposed here.

II The design of an optimal transportation network for a given land-use plan

This section presents a model for designing an optimal transportation network, for a new town for instance, when congestion is assumed on the links of the network. Although it is more realistic to assume that the real world design problem is discrete[1], discrete approaches could not handle problems with several thousand links. Thus one has to use optimization models where link capacity can vary continuously. Such a model was stated by Steenbrink (1974, p.91). A decomposition approach was presented by him to solve the problem, resulting in a master problem for which he proposed a heuristic method. Newell (1974) stated an analogous model and suggested that a decomposition approach might solve it. Dantzig *et al.* (1976) showed that Steenbrink's master problem could be solved exactly by traffic assignment algorithms with congestion and fixed demand for travel such as are reported by Nguyễn (1974) or LeBlanc *et al.* (1975). Abdulaal and LeBlanc (1976) have independently proposed the same method as Dantzig *et al.* Dantzig, Maier and Lansdowne (1976) stated that this approach incorporates the following features: "continuous investment decision variables; system optimal traffic assignment; the total travel time on a link is a convex function of link flow and investment; a convex investment cost function." It will be shown in this section that, if the preceding features are restricted by requiring that the investment cost function be linear and that the volume-delay function satisfy some assumptions that are met in particular by the BPR volume-delay function, the network design problem thus formulated

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[1] Cf. Billheimer and Gray (1973), Boyce *et al.* (1973), Dionne and Florian (1977), LeBlanc (1975), Los (1976b) for discrete approaches to network design.

reduces to an all-or nothing traffic assignment problem. The main advantage of this approach is that very efficient shortest path codes exist to solve this traffic assignment problem (see, for example, Dial *et al.*, 1977).

Mathematical formulation of the problem and assumptions

We are given a maximal network containing a set A of potential links a. A subset of the set of nodes of this network are origins or destinations for trips. The land-use plan is assumed known and the demand for travel between origin p and destination q is a constant D_{pq}; D_{pq} subdivides among path flows $h_{m,pq}$ where m is a path between p and q; v_a is the flow of vehicles or persons on link a.(In this paper, we will assume that there is a known constant and uniform car occupancy factor, so that the flows and demands can without ambiguity be measured in vehicles or in person-trips).

z_a is the practical capacity on link a. The travel time on link a is a function $s_a(v_a, z_a)$ of the link flow and of the link capacity; ℓ_a is the investment cost per unit of practical capacity on link a ; λ is a known positive multiplier transforming dollars into time units and depends on a social rate of discount and on a value of travel time. The network design problem may be stated as the following minimization problem, called NDP (Network Design Problem) in the rest of the paper:

(NDP):

$$\min_{h_{m,pq},\, z_a} \quad F = \sum_{a \in A} v_a\, s_a\,(v_a, z_a) + \lambda \sum_{a \in A} \ell_a\, z_a \tag{1}$$

$$\text{s.t.} \quad \sum_m h_{m,pq} = D_{pq} \qquad \text{all } p,q \tag{2}$$

$$h_{m,pq} \geq 0 \tag{3}$$

$$z_a \geq 0 \tag{4}$$

where $v_a = \sum_p \sum_q \sum_m \delta_{am,pq} \, h_{m,pq}$ all a

and $\delta_{am,pq} = \begin{cases} 1 & \text{if link a belongs to path m between p and q} \\ 0 & \text{otherwise} \end{cases}$

The objective function (1) represents the tradeoff between travel time and capital costs. Constraint (2) is a conservation of flows constraint. The links for which z_a would be zero at the optimum would not be built. This model would therefore decide on the topology of the optimal network as well as on the optimal capacities of the arcs that should be built. It is therefore appropriate for a completely new transportation network.

Assumptions implicit in this model

N1 : The arc capacities can vary continuously

N2 : For a given network the flows are given by a system-optimal pattern.

N3 : The investment cost function for the network links is assumed linear.

The congestion function meets the following assumptions:

N4 : For each fixed $v_a \geq 0$, $s_a(v_a, z_a)$ is convex in z_a for $z_a \geq 0$. For each fixed $z_a \geq 0$, $s_a(v_a, z_a)$ is convex in v_a for $v_a \geq 0$.

N5 : There exists a function σ_a such that $s_a(v_a, z_a) = \sigma_a(u_a)$ with

$u_a = \frac{v_a}{z_a}$ for $z_a > 0$ and $v_a \geq 0$ with $\sigma_a(u_a)$ convex and strictly increasing for $u_a > 0$.

$s_a(0,0) = \sigma_a(\bar{u}_a)$ where $\bar{u}_a$ is the unique [1] solution to the equation:

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[1] It will be proved below that there is one and only one solution to this equation.

$$u_a^2 \frac{d\sigma_a}{du_a} = \lambda \ell_a$$

We define a congestion function that can be shown to meet assumptions N4 and N5 (Cf. Appendix). It is identical to the BPR function for $z_a > 0$ and $v_a \geq 0$.

$$s_a(v_a, z_a) = t_a \left[1 + \rho \left(\frac{v_a}{z_a}\right)^{\mu}\right] \text{ for } z_a > 0 \text{ and } v_a \geq 0$$

$$s_a(v_a, z_a) = d_a \text{ for } z_a = 0 \text{ and } v_a = 0 \text{ with}$$

$$d_a = t_a + (\lambda \ell_a)^{\frac{\mu}{\mu+1}} (\rho t_a)^{\frac{1}{\mu+1}} \mu^{-\frac{\mu}{\mu+1}}$$

t_a is the free-flow travel time, assumed given for each potential arc; ρ and μ are known positive constants with $\mu \geq 1$.

N6 : Only one mode of transportation, namely the private automobile, is assumed in this model.

N7 : The topology ot the maximal transportation network is given. Only the capacities of the arcs can vary. Arc capacities can be zero, because the problem consists in constructing a new network and not in improving an existing network.

Solution method

Several results have to be proved before proposing a solution method.

Theorem 1 : The solution $\bar{u}_a$ to the equation

$$u_a^2 \frac{d\sigma_a}{du_a} = \lambda \ell_a \qquad (5)$$

exists and is unique.

Proof : From N5 we know that $\frac{d\sigma_a}{du_a} > 0$ for $u_a > 0$ and $\frac{d^2\sigma_a}{du_a^2} \geq 0$ for $u_a > 0$.

The derivative of $u_a^2 \frac{d\sigma_a}{du_a}$ with respect to u_a is

$u_a^2 \frac{d^2\sigma_a}{du_a^2} + 2u_a \frac{d\sigma_a}{du_a}$ and is always strictly positive for $u_a > 0$.

Therefore $u_a^2 \frac{d\sigma_a}{du_a}$ is a strictly monotone increasing function for $u_a > 0$, such that $u_a^2 \frac{d\sigma_a}{du_a} =)$ for $u_a = 0$ and

$$\lim_{u_a \to \infty} (u_a^2 \frac{d\sigma_a}{du_a}) = \infty .$$

Therefore equation (5) has a unique solution for all $\lambda > 0$.

Corrolary : $s_a(0,0) = \sigma_a(\bar{u}_a)$ is uniquely defined and by convention we set

$$\lim_{\substack{v_a \to 0+ \\ z_a \to 0+}} \frac{v_a}{z_a} = \bar{u}_a \quad . \tag{6}$$

Theorem 2 : For a solution to be optimal we must have, for all links a, either $v_a > 0$ and $z_a > 0$ or $v_a = 0$ and $z_a = 0$.

Proof : We cannot have $z_a > 0$ and $v_a = 0$ at the optimum because the objective function could be diminished simply by setting $z_a = 0$ for all links a for which $v_a = 0$. And we cannot have $v_a > 0$ and $z_a = 0$ at the optimum because, according to N5

$$\lim_{z_a \to 0+} s_a(v_a, z_a) = \lim_{u_a \to \infty} \sigma_a(u_a) = \infty$$

and therefore F would have an infinite value.

In order to solve (NDP) we will neglect constraint (4). This is justified by the fact that the optimal solution to the problem (1), (2) and (3) will be shown to satisfy also constraint (4); z_a therefore disappears from the constraint set. In addition, from assumption (N4) and from the fact that the sum of convex functions is convex we know that a necessary and sufficient condition for z_a to be optimal is:

$$\frac{\delta F}{\delta z_a} = -\left(\frac{v_a}{z_a}\right)^2 \frac{d\sigma_a}{du_a} + \lambda \ell_a = 0 \quad \text{all } a \tag{7}$$

$$\text{or} \quad u_a^2 \frac{d\sigma_a}{du_a} = \lambda \ell_a \quad \text{all } a \tag{8}$$

From theorem 1, we know that (8) has a unique solution $\bar{u}_a$. Since, at the optimum, we have either $v_a > 0$ and $z_a > 0$ or $v_a = 0$ and $z_a = 0$ and since we made the convention that

$$\lim_{\substack{z_a \to 0+ \\ v_a \to 0+}} \frac{v_a}{z_a} = \bar{u}_a$$

the following relationships

$$v_a = \bar{u}_a z_a \tag{9}$$

$$s_a(v_a, z_a) = \sigma_a(\bar{u}_a) \tag{10}$$

hold at the optimum.

For the BPR function, equation (8) becomes

$$\mu t_a \rho \, v_a^{\mu+1} \, z_a^{-\mu-1} = \lambda \ell_a \tag{11}$$

and therefore

$$\left(\frac{v_a}{z_a}\right)^{\mu+1} = \frac{\lambda \ell_a}{\mu t_a \rho} \tag{12}$$

$$\text{or} \quad \bar{u}_a = \left(\frac{\lambda \ell_a}{\mu t_a \rho}\right)^{\frac{1}{\mu+1}} \tag{13}$$

$$d_a = \sigma_a(\bar{u}_a) = t_a + \rho t_a \, \bar{u}_a^{\mu} \tag{14}$$

$$\text{or} \quad d_a = t_a + (\lambda \ell_a)^{\frac{\mu}{\mu+1}} (\rho t_a)^{\frac{1}{\mu+1}} \mu^{-\frac{\mu}{\mu+1}} \tag{15}$$

With the travel times given by (15) or more generally by (10), one can find the optimal $h_{m,pq}$'s by solving the following master problem:

$$\underset{h_{m,pq}}{\text{Min}} \quad F = \sum_a \theta_a \, v_a \tag{16}$$

$$\text{s.t.} \quad \sum_m h_{m,pq} = D_{pq} \qquad \text{all } p,q \tag{17}$$

$$h_{m,pq} \geq 0 \tag{18}$$

$$\text{where } v_a = \sum_p \sum_q \sum_m \delta_{am,pq} \, h_{m,pq} \qquad \text{all a}$$

$$\text{and} \quad \theta_a = \sigma_a(\bar{u}_a) + \lambda \ell_a \, \bar{u}_a^{-1} \qquad \text{all a} \tag{19}$$

θ_a can be interpreted as a weighted sum of the travel time per unit of flow and of the cost (in time units) of the investment in the capacity of link a necessary per unit of flow at the

optimum ratio $\bar{u}_a$. The problem (16), (17) and (18) is a traffic assignment problem with constant arc costs which can be solved very efficiently by many shortest path codes (see, for example, Dial *et al.*, 1977). Once the shortest paths between origin-destination pairs have been computed on the basis of the costs θ_a, the optimal link flows v_a are given by

$$v_a^* = \sum_p \sum_q \delta'_{a,pq} D_{pq} \tag{20}$$

where $\delta'_{a,pq} = \begin{cases} 1 & \text{if a is on a shortest path between p and q} \\ 0 & \text{otherwise} \end{cases}$

and the optimal capacities are given by

$$z_a^* = \bar{u}_a^{-1} v_a^* \tag{21}$$

These capacities cannot be negative because the v_a^* cannot be negative and therefore these flows and capacities solve the original problem (NDP) and constraint (4) was indeed redundant.

Numerical experiments

This method was applied to the determination of the optimal network for the City of Hull (Quebec, Canada). The existing road network was taken as the maximal network and the problem was to determine the optimal capacities assuming the network had never been built. The travel demand was assumed fixed and equal to the actual morning peak-hour demand. The number of potential links was 798. The number of nodes in the network was 501. The number of centroids (i.e. nodes that can attract or generate traffic) was 23. The value of time chosen was \$3.00/hour and the capital cost per vehicle/hour and per mile was taken to be \$500.00 The BPR function was chosen for the congestion function and the free-flow travel times on the links were fixed at their actual values. Table 1

Characteristics of the optimal network / Social rate of discount	Numbers of links that are built	Total length in miles	Minimal link capacity in veh/hr	Maximal link capacity in veh/hr	Average link capacity in veh/hr	Total travel time in hours	Total capital cost in $	Objective function in hours	Objective function in hours with a user-equilibrium
.01	155	18.44	19.1	5,065.6	1,114.9	238,240	11,602,000	276,530	276,665
.02	167	19.65	27.4	4,473.8	900.7	244,380	10,197,000	311,020	311,295
.05	164	19.26	24.9	3,807.2	782.56	260,160	8,700,600	398,270	398,590
.10	154	18.48	12.9	3,384.9	740.5	281,440	7,713,600	515,180	515,390
.15	159	18.63	12.1	3,169.9	680.22	300,030	7,238,100	614,730	614,835
.20	157	18.56	11.6	3,031.7	647.81	314,470	6,913,700	698,570	698,820
.50	160	19.05	10.2	2,672.7	552.53	379,930	6,083,100	1,055,800	1,056,335
Results for the current network:	Total travel time (in hours) with a user-equilibrium traffic assignment algorithm: 385,535								
	Total capital cost in CAN$: 13,146,000								

Table 1 : Results obtained by the optimal network design algorithm for the Hull network

reports the results obtained for different values of the social rate of discount. One can observe very clearly that the optimal solution depends on the tradeoff between capital costs and user costs, which is itself a function of the social rate of discount and that the existing network is far from being optimal at any social rate of discount. The shortest path code used is due to Pappe and is reported in Dial *et al.* (1977). The execution time varied from 10.59 seconds to 17.97 seconds on the CDC Cyber 173 of the Université de Montréal. The last column of table 1 was obtained by computing the value of the objective function when a system optimal traffic equilibrium is replaced by a user equilibrium. The resulting total travel times are nearly the same. This means that for this particular formulation of the network design problem, the assumption of a system-optimal trip pattern gives results that are nearly the same as those that would have been obtained by formulating the network design problem under the assumption of a user-equilibrium.

III The design of an optimal land use plan

This section presents a model for designing an optimal land-use plan for given travel costs between location pairs. The assumption of fixed travel costs is equivalent to the assumption of a fixed transportation network with no congestion. It is a discrete model with an objective function that represents the tradeoff between site costs and total travel cost. It is an extension of the Koopmans-Beckmann (1957) problem. Computational results for the latter problem have been reported in Los (1977). A heuristic technique of the hill-climbing type, called ACTIV, was shown to be extremely good if solution quality and computational efficiency are

both taken into account: ACTIV took 89.32 seconds on a CDC Cyber 173 to reach a local optimum for a problem of size n = 72. The time complexity is $O(n^3)$ and the storage space is $(8n^2 + 4n)$ words. The Koopmans-Beckmann problem was previously proposed by Los (1975, 1978) to solve the land-use planning problem. It offered a simple way of taking into account the interactions between the location decisions of different activities in a combinatorial programming approach. However, as emphasized by Hopkins (1977) it assumes that the demand for travel is fixed for each activity pair, which is a very strong and unrealistic assumption. In reality, the number of trips between activity pairs is a function of the relative distance (or more generally, the transportation cost) to substitutable activities, as is well known by urban transportation planners. In addition, total demand for travel of a given type (shopping trips say) is elastic but this aspect will not be considered here. Hopkins (1977) proposes an approach that distinguishes between demand activities (such as residences) and central facilities (such as schools) and uses iteratively an algorithm for a location-allocation model and an algorithm to solve the quadratic assignment problem. This section will formalize mathematically this land-use design problem and will present a method to solve this problem that does not require a location-allocation algorithm but only an algorithm for solving the quadratic assignment problem.

Mathematical formulation of the problem and assumptions

We have to assign N discrete activities to N discrete locations. The set of activities is divided into two subsets: the "demand" activities $\{1,2,\ldots,n\}$ have fixed interactions among themselves and the "facilities" $\{n+1,\ldots,N\}$ have fixed interactions among themselves but are such that the interactions

between "demand" activities and activities called facilities can be zero or a given constant. Typically an activity of the first type would be a residence unit or a factory, while an activity of the second type would be a school, a shopping center, or a recreation center. For a given residence unit all the school trips would be destined to the nearest school. There are N-n facilities divided among H types h. Each demand activity is "allocated" to one and only one facility of a given type. The problem to solve is to assign the N activities to the N locations and to allocate the n demand activities among the (N-n) facilities, so as to minimize an objective function which represents a tradeoff between site costs and transportation user costs. The assignment of activities to locations is represented by the matrix:

$$Y = [y_{ip}] \qquad \begin{matrix} i = 1,\ldots,N \\ p = 1,\ldots,N \end{matrix}$$

where $$y_{ip} = \begin{cases} 1 \text{ if activity } i \text{ is assigned to location } p \\ 0 \text{ otherwise} \end{cases}$$

The allocation of demand activities to facilities is represented by the matrix:

$$X = [x_{if}] \qquad \begin{matrix} l = 1,\ldots,n \\ f = n+1,\ldots,N \end{matrix}$$

where $$x_{if} = \begin{cases} 1 \text{ if the number of trips between } i \text{ and } f \text{ is non zero} \\ 0 \text{ otherwise} \end{cases}$$

$$B = [b_{ip}] \qquad \begin{matrix} i = 1,\ldots,N \\ p = 1,\ldots,N \end{matrix}$$

is the matrix of site costs; b_{ip} is the site cost of assigning activity i

to location p; it can represent the sum of a development cost, of a spatial congestion cost preventing building above a certain density limit, and an ecological cost taking into account the amenity or disamenity characteristics of assigning a particular activity to a particular location.

$$Q = [q_{ij}] \qquad \begin{matrix} i = 1,\ldots,n \\ j = 1,\ldots,n \end{matrix}$$

is the matrix of interactions (i.e. trips) between the demand activities

$$R = [r_{fg}] \qquad \begin{matrix} f = n+1,\ldots N \\ g = n+1,\ldots,N \end{matrix}$$

is the matrix of interactions between the facilities.

Let H be the set of indexes corresponding to the facilities of type h ($h=1,\ldots,H$)

$$K = [K_{ih}] \qquad \begin{matrix} i = 1,\ldots,n \\ h = 1,\ldots,H \end{matrix}$$

is the matrix of interactions between demand activities and facilities. More precisely, K_{ih} is the number of trips between demand activity i and one facility of type h.

$$C = [c_{pq}] \qquad \begin{matrix} p = 1,\ldots,N \\ q = 1,\ldots,N \end{matrix}$$

is the matrix of transportation costs between location pairs. The matrices Q, R and K refer to a given unit of time (the day for instance). We require Y and X to be the solution to the following minimization problem called LUDP (Land Use Design Problem):

(LUDP):

$$\text{Min } G(Y,X) = \sum_{i=1}^{N}\sum_{p=1}^{N} b_{ip}\, y_{ip}$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{N}\sum_{q=1}^{N} q_{ij}\, c_{pq}\, y_{ip}\, y_{jq}$$

$$+ \sum_{f=n+1}^{N}\sum_{g=n+1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N} r_{fg}\, c_{pq}\, y_{fp}\, y_{gq}$$

$$+ \sum_{i=1}^{n}\sum_{h=1}^{H}\sum_{f\in H}\sum_{p=1}^{N}\sum_{q=1}^{N} K_{ih}\,\overline{x_{if}}\, c_{pq}\, y_{ip}\, y_{fq} \qquad (22)$$

$$\text{s.t.} \quad \sum_{i=1}^{N} y_{ip} = 1 \qquad p = 1,\ldots,N \qquad (23)$$

$$\sum_{p=1}^{N} y_{ip} = 1 \qquad i = 1,\ldots,N \qquad (24)$$

$$\sum_{f\in H} x_{if} = 1 \qquad \begin{matrix} i = 1,\ldots,n \\ h = 1,\ldots,H \end{matrix} \qquad (25)$$

$$y_{ip} = \begin{cases} 1 \\ 0 \end{cases} \qquad x_{if} = \begin{cases} 1 \\ 0 \end{cases}$$

Constraint (23) states that each location is the site of one and only one activity. Constraint (24) states that each activity is assigned to one and only one location. Constraint (25) states that all trips between a given demand activity and facilities of a given type occur between the given demand activity and only one facility of the given type. For a given matrix X, (LUDP) reduces to a quadratic assignment problem of the Koopmans-Beckmann type and for a given Y the problem is decomposable into nH trivial subproblems: each subproblem requires the allocation of a demand activity to the closest facility of a given type. (LUDP) is thereby shown

to be an extension of the Koopmans-Beckmann problem.

Assumptions implicit in this model

LU1 : Land use is treated in a discrete fashion. The size of each activity must be determined before applying the model: a given land use, say open space, must be divided into a given number of discrete modules.

LU2 : The model (LUDP), as stated here, focuses on accessibility, development costs and ecological costs but only insofar as the latter are independent of the relative location of the facilities. In particular we assume in this formulation that there are no spillover effects such as air pollution. The model however can be easily extended to cover such effects.

LU3 : This model makes the assumption that all trips that occur between a demand activity and facilities of a given type are allocated to one and only one facility of that type. This is a very simple way but a very rigid one of introducing variable demand for travel in the Koopmans-Beckmann model.

LU4 : There are no explicit capacity constraints on the facilities, i.e. there is no limit to the total number of trips that can be destined to a given facility.

Solution method

No efficient exact algorithm has yet been devised to solve the Koopmans-Beckmann problem (see, for example, Los, 1977). A *fortiori* no efficient exact method exists to solve the model (LUDP),which is itself an extension

of the Koopmans-Beckmann problem. One has therefore to resort to a heuristic technique. The algorithm (QUALF) proposed here is a hill-climbing technique which solves iteratively for Y, X being temporarily fixed, and vice-versa. More precisely QUALF works as follows:

a. Start with an initial assignment of activities to locations obtained by random generation for instance. (An alternative method of generating an initial plan would consist in randomly locating central facilities and then in solving the linear assignment problem obtained by neglecting fixed interactions, i.e. by considering only central facility interactions in the model [1].

b. Allocate each demand activity to the nearest facility of each type.

c. Solve the resulting quadratic assignment problem by ACTIV [2].

d. Iterate on steps b and c until no further improvement is possible.

Several local optima can be generated by choosing different initial assignments of activities to locations at step a.

Numerical experiments

QUALF was implemented on the CDC Cyber 74 of the Université de Montréal and applied to test problems of different sizes. Site costs were set equal to zero and only interactions between demand activities and facilities were non-zero. Table 2 shows the variation of the execution time with N. QUALF always improved on the initial cost by about 50%. A more detailed presentation of these experiments can be found in Los (1976a). The efficiency of QUALF is only limited by the efficiency of its quadratic assignment component. Therefore it can be used to solve problems of size up to N=72. For very large size problems it is necessary to use a decomposition approach

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[1] This was suggested to me by L. Hopkins.

[2] Cf. Los (1977) for a detailed description and evaluation of that method.

such as was used by Hopkins (1977).

Problem Size: N	Number of Local Optima Generated per problem [1]	Range of Variation for the Execution Times in Seconds of CDC Cyber 74
16	6	.08 to .15
25	10	.30 to .59
36	16	.92 to 2.14
49	20	2.36 to 5.12

Table 2: Results of numerical experiments performed with QUALF (see Los, 1976a)

IV The simultaneous optimization of land-use and transportation

In this section the problem consisting in optimizing simultaneously a transportation network and a land use plan is formalized. The resulting optimization model is a synthesis of the network design model and of the land-use design model presented in the two preceding sections. We have to assign discrete activities to discrete locations which are a subset of the set of nodes of a given maximal transportation network, and to determine the capacities of the arcs of the network (Figure 1). As in section III, activities will be either "demand" activities or facilities and we assume again that each demand activity is "allocated" to one and only one facility of each type. The problem to solve is to assign the N activities to the N locations, to allocate the n demand activities among the (N-n) facilities and to choose the practical capacities of the arcs of the network, so as to minimize a total cost composed of site costs, capital costs for the links of the network and transportation user costs, under the assumption that there is transportation

[1] One problem was considered for each problem size.

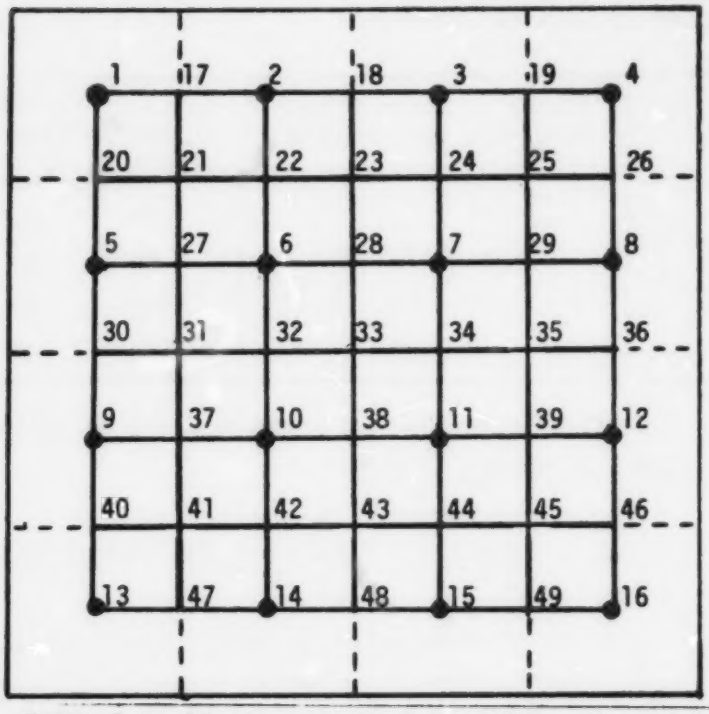

Figure 1: Nodes 1 to 16 are potential locations of activities, while nodes 17 to 49 are nodes on the network that do not generate or attract traffic.

congestion. Y, X, B, Q, R, K , v_a, z_a, ℓ_a, $h_{m.pq}$, $s_a(v_a, z_a)$ have been defined in sections II and III. The joint optimization problem, referred to as P below, consists in finding the values of the variables y_{ip}, x_{if}, z_a and $h_{m,pq}$ so as to minimize

$$\phi = \sum_{i=1}^{N} \sum_{p=1}^{N} b_{ip}\, y_{ip} + \sum_{a \in A} v_a s_a(v_a, z_a) + \sum_{a \in A} \ell_a z_a \qquad (26)$$

s.t.

$$\sum_{m} h_{m,pq} = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}\, y_{ip}\, y_{jq} + \sum_{f=n+1}^{N} \sum_{g=n+1}^{N} r_{fg}\, y_{fp}\, y_{gq}$$

$$+ \sum_{i=1}^{n} \sum_{h=1}^{H} \sum_{f \in H} K_{ih}\, x_{if}\, y_{ip}\, y_{fq} \qquad (27)$$

$$p = 1, \ldots, N$$

$$q = 1, \ldots, N$$

$$\sum_{i=1}^{N} y_{ip} = 1 \qquad p = 1,\ldots,N \tag{28}$$

$$\sum_{p=1}^{N} y_{ip} = 1 \qquad i = 1,\ldots,N \tag{29}$$

$$\sum_{f \in H} x_{if} = 1 \qquad \begin{matrix} i = 1,\ldots,n \\ h = 1,\ldots,H \end{matrix} \tag{30}$$

$$y_{ip} = \begin{cases} 1 & \text{if activity } i \text{ is assigned to location } p \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

$$i = 1,\ldots,N$$

$$p = 1,\ldots,N$$

$$x_{if} = \begin{cases} 1 & \text{if demand activity } i \text{ is allocated fo facility } f \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

$$i = 1,\ldots,n$$

$$g = n+1,\ldots,N$$

$$h_{m,pq} \geq 0 \tag{33}$$

$$z_a \geq o \tag{34}$$

where $v_a = \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{m} \delta_{am,pq} \, h_{m,pq}$ all a

$$\delta_{am,pq} = \begin{cases} 1 & \text{if link a belongs to path m between nodes p and q} \\ 0 & \text{otherwise} \end{cases}$$

The first term of (26), $\sum_{i=1}^{N} \sum_{p=1}^{N} b_{ip} \, y_{ip}$, represents the sum of site costs; the second term, $\sum_{a \in A} v_a \, s_a \, (v_a, z_a)$, represents the total transportation user cost; the third term, $\sum \ell_a \, z_a$ represents the construction cost of the transportation network. Thus this objective function represents the trade-off between capital costs and user accessibility. A multiplier λ char-

acterizes the tradeoff between capital costs and user costs and is a function of the social rate of discount and of the value of time. It has been omitted from the formulation of (P) but it is present implicitly. If we were given a budget constraint instead of a value of λ, by a Lagrange multiplier technique we could introduce the investment cost in the objective function and solve a series of problems for different values of λ. (27) is a conservation of flows constraint; (28) states that there is one and only one activity per location; (29) states that there is one and only one location for each activity; (30) states that all trips between a given "demand" activity and facilities of a given type occur between the given demand activity and only one facility of the given type. For given X and Y, (P) reduces to the network design problem of section II. (P) differs from the land-use design problem of section III by the fact that travel costs are no longer constant but are dependent on the chosen link capacities and on the location of the activities as well as on the allocation of demand activities to facilities. The assumptions N1 to N7 and LU1 to LU4 still hold in the formulation of problem (P).

Solution method

Problem (P) belongs to the class of problems called discrete-convex programming by Geoffrion (1975). We introduce the following notation:

$$\delta = (\ldots, y_{ip}, \ldots, x_{if}, \ldots)$$

$$w = (\ldots, h_{m,pq}, \ldots, z_a, \ldots)$$

$$C_\delta = \sum_{i=1}^{N} \sum_{p=1}^{N} b_{ip}\, y_{ip}$$

$$F(w) = \sum_{a \in A} v_a\, s_a\,(v_a, z_a) + \sum_{a \in A} \ell_a\, z_a$$

Let Δ be the set of constraints (28), (29), (30), (31), (32) and W_δ be the constraint set (27), (33) and (34). (P) can be restated as:

$$\begin{aligned} &\underset{\delta, w}{\text{Min}} \quad \phi = C_\delta + F(w) \\ (DC) \qquad &\text{s.t.} \quad \delta \in \Delta \\ &\text{and} \quad w \in W_\delta \end{aligned}$$

Problem (DC) involves optimization over the joint space of the δ and w variables. It is equivalent to the following problem (D), defined as its "projection" onto the space of the δ variables (see, for example, Geoffrion, 1972, theorem 1; Geoffrion, 1975).

$$(D) \qquad \underset{\delta \in \Delta}{\text{Minimize}} \; [C_\delta + \underset{w \in W_\delta}{\text{Minimum}} \; F(w)]$$

The inner minimization problem can be solved very simply. It has been shown in section II of this article that (P1) reduces to (P2)

(P1) : Find $w = (h_{m,pq}$ and $z_a)$ so as to minimize

$$F(w) = \sum_{a \in A} v_a \, s_a \, (v_a, z_a) + \sum_{a \in A} \ell_a \, z_a$$

$$\text{s.t.} \quad w \in W_\delta$$

(P2) : Find $h_{m,pq}$ so as to minimize

$$\sum_a \theta_a \, v_a$$

$$\text{s.t.} \quad (27) \text{ and } (33)$$

$$\text{with } \theta_a = \sigma_a \, (\bar{u}_a) + \ell_a \, \bar{u}_a^{-1} \qquad (35)$$

where $\bar{u}_a$ is the unique solution to

$$u_a^2 \, \frac{d\sigma_a}{du_a} = \ell_a \qquad (36)$$

(P2) is a traffic assignment problem with constant arc costs θ_a which can be solved by a shortest path code. The demand between origin-destination pairs can be computed for known y_{ip}'s and x_{if}'s, hence the flows v_a; and the optimal capacities on the arcs are then given by:

$$z_a = \bar{u}_a^{-1} v_a.$$

It follows from (P2) that $\underset{w \in W_\delta}{\text{Minimum}} F(w)$ can be rewritten as:

$$\sum_{p=1}^{N} \sum_{q=1}^{N} c_{pq}(\theta) \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}\, y_{ip}\, y_{jq} + \sum_{f=n+1}^{N} \sum_{g=n+1}^{N} r_{fg}\, y_{fp}\, y_{gq} \right.$$

$$\left. + \sum_{i=1}^{n} \sum_{h=1}^{H} \sum_{f \in H} K_{ih}\, x_{if}\, y_{ip}\, y_{fq} \right\} \qquad (37)$$

where $c_{pq}(\theta)$ is the cost of a shortest path between p and q when the arc costs are given by the vector

$$\theta = (\theta_1, \ldots, \theta_a, \ldots), \quad a \in A.$$

The projected problem (D) therefore becomes

$$(D) \quad \underset{y_{ip}, x_{if}}{\text{Min}} \sum_{i=1}^{N} \sum_{p=1}^{N} b_{ip}\, y_{ip} + \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}\, c_{pq}(\theta)\, y_{ip}\, y_{jq}$$

$$+ \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{f=n+1}^{N} \sum_{g=n+1}^{N} r_{fg}\, c_{pq}(\theta)\, y_{fp}\, y_{gq}$$

$$+ \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{i=1}^{n} \sum_{h=1}^{N} \sum_{f \in H} K_{ih}\, c_{pq}(\theta)\, x_{if}\, y_{ip}\, y_{fq} \qquad (38)$$

s.t. (28), (29), (30), (31) and (32).

(D) is equivalent to (LUDP) of section III and can be solved by a heuristic algorithm such as QUALF.

From this discussion follows the method for solving (P).

1. For each arc a, find the unique solution $\bar{u}_a$ to equation (36), i.e. the optimal ratio of flow to capacity. For the particular case of the BPR function, $\bar{u}_a$ is given by

$$\bar{u}_a = \left(\frac{\ell_a}{\mu t_a \rho}\right)^{\frac{1}{\mu+1}} \tag{39}$$

2. Compute the arc costs θ_a by (35). For the BPR function:

$$\theta_a = t_a + \rho t_a \bar{u}_a^{\mu} + \ell_a \bar{u}_a^{-1} \tag{40}$$

3. Compute the shortest path costs $c_{pq}(\theta)$.

4. Solve the projected problem (D). This yields the optimal values (or at least near-optimal if a heuristic technique like QUALF is used) of y_{ip} and x_{if} : y^*_{ip} and x^*_{if}.

5. The optimal (or near-optimal) flows v_a are given by

$$v^*_a = \sum_{p=1}^{N} \sum_{q=1}^{N} \delta'_{a,pq} D_{pq} \tag{41}$$

where $\delta'_{a,pq} = \begin{cases} 1 & \text{if a is on a shortest path (in the sense of the } \theta \text{ costs) between p and q} \\ 0 & \text{otherwise} \end{cases}$

and $D_{pq} = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} y^*_{ip} y^*_{jq} + \sum_{f=n+1}^{N} \sum_{g=n+1}^{N} r_{fg} y^*_{fp} y^*_{gq}$

$$+ \sum_{i=1}^{n} \sum_{h=1}^{H} \sum_{f \in H} K_{ih} x^*_{if} y^*_{ip} y^*_{fq} \tag{42}$$

6. The optimal (or near-optimal) capacities on the arcs are then given by:

$$z_a^* = \bar{u}_a^{-1} v_a^*$$

The only steps that require special algorithms are step 3 and step 4. Step 3 requires a shortest path code and a good candidate for such a code is Pappe's code that was used in section II to solve (NDP). Step 4 requires an algorithm to solve the combinatorial programming problem (LUDP) and it was shown in section III that QUALF provided a good heuristic to solve it. Steps 1, 2, 5 and 6 do not require any special purpose algorithms. Therefore the simultaneous optimization of land-use and transportation as formulated by (P) reduces computationally to the addition of a network design problem like (NDP) to a land-use design problem like (LUDP).

V Discussion of the assumptions of the model

We discuss in turn the relevance of each of the assumptions of model (P) and examine how their limitations can be overcome.

N1 The assumption that arc capacities vary continuously is only a first approximation. It is more realistic to assume that the real-world network design problem is discrete: for instance the number of lanes is an integer number and the width of the lanes does not vary at will. However a discrete approach could not handle a problem with several thousand links. The impracticality of obtaining solutions could justify in itself the assumption of continuous arc capacities. Besides it is often conjectured by urban and transportation planners (see, for example, Boyce *et al.* , 1970) that the performance of a land-use-transportation plan is more sensitive to the land-use decisions than to the network decisions. This conjecture would justify treating the network design problem in a less refined way than the land-use design problem in a joint optimization model.

N2 : The assumption of a system-optimal flow pattern is also only an approximation made to render the formulation of the design problem computationaly feasible. If the real traffic assignment were a user-equilibrium[1], the system-optimal formulation could be used to find a lower and an upper bound to the true optimum. Let f^* be the true optimum, ϕ^* be the optimum of problem (P) (this implicitly assumes that an exact technique is used to solve (D)) and let ϕ^u be the total cost of the optimal solution of (P), but computed under the assumption of a user-equilibrium, instead of a system-optimal pattern. Then we have:

$$\phi^* \leq f^* \leq \phi^u .$$

If the interval $[\phi^*, \phi^u]$ is sufficiently small the assumption N2 is not restrictive. Indeed this property was observed in the numerical experiments of section II on the Hull network.

N3 : The assumption of a linear investment cost function is a compromise between the assumption of convexity and the assumption of concavity. This compromise seems justified by Steenbrink according to whom the investment cost function is piecewise linear and is concave first and then convex (Cf. Steenbrink, 1974, p. 219). A possible linear approximation to the investment cost function of figure 2 might be the dotted line.

Assumptions N4 and N5 about the congestion function are justified by the fact that they are met by the BPR function, which has been shown to

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[1] Cf. Florian (1976), LeBlanc *et al.* (1975) and Nguyen (1974) for a detailed discussion of traffic assignment concepts and solution algorithms.

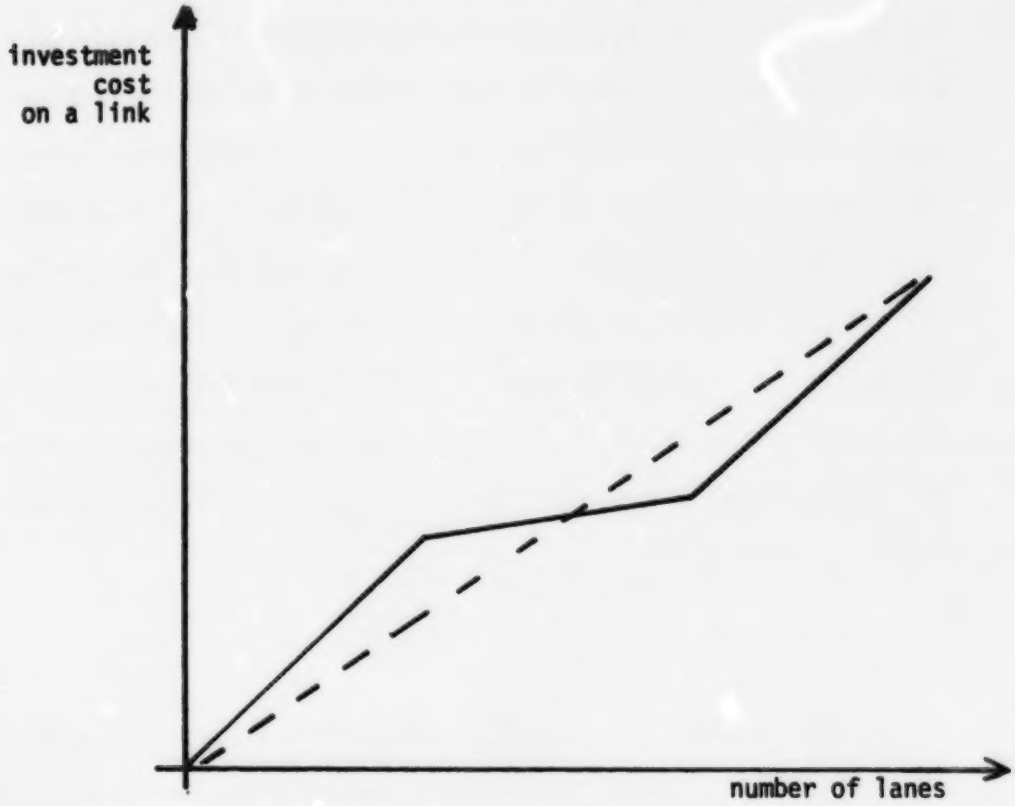

Figure 2: Investment cost function

enable a good approximation to more complex congestion functions (see, for example, Branston, 1976).

N6 : The one-mode assumption is a strong limitation of the present model which should eventually be relaxed for policy purposes. This however will require much more research. Multimodal equilibrium (e.g. equilibrium with buses and private automobiles) for exogenous supply decisions is in itself a non-trivial problem: in particular modal split depends on congestion and on traffic management assumptions (such as the possible existence of special bus lanes). See Florian

and Nguyen, 1977, and Florian 1977, for instance). Besides, a variety of decisions (bus routes, frequency, etc.) have to be included in an optimal supply model for public transit (see Siv Scheele, 1977). For all these reasons the optimization of a multimodal transport system is not an easy problem to formalize and to solve, *a fortiori* its combination with land-use planning.

N7 : The assumption that the topology of the maximal transportation network is given is not a very serious limitation. This comes from the fact that the number of links and the number of nodes in the model proposed in this paper influence only the computation time of the shortest path algorithm. Very efficient codes exist to solve that problem so that many intermediate nodes can be *a priori* introduced in the maximal network.

LU1: Treating land use in a discrete fashion is more realistic than treating it in a continuous fashion because of economies of scale and minimum size requirements for some activities. Dummy interactions can be introduced to force contiguity between activity modules that belong to the same real activity.

LU2: The present formulation, focusing on capital costs and transportation costs is not necessary to keep the structure of the model. There would be no difficulty at all in including air pollution interactions or incompatibility among land uses in the objective function. This would only add to the objective function new quadratic terms that would be independent of the transportation network. The only change in the

method proposed here would consist in modifying step c of QUALF to replace ACTIV by another heuristic which is similar to ACTIV except that it assumes multiple types of interactions instead of only one type of interaction. QUALF would still remain computationally tractable since the execution time of this generalized version of ACTIV can be shown to increase linearly with the number of interaction types[1].

LU3: The assumption that all "central facility" trips of a given type are allocated to one and only one facility of that type is more relevant to certain trips (e.g. school trips) than to others (e.g. shopping trips). The latter are "partially substitutable" trips, which means that the interactions between a given residential activity and each shopping center, for instance, decline with distance (or more generally transportation cost) but not in an all-or-nothing fashion (see, for example, Harris, 1964). This simplifying assumption is made for computational purposes and it is conjectured that it would not influence too much the optimal plan. This conjecture is all the more valid if the trip deterrence parameter of the exponential cost function is assumed large.

LU4: The absence of capacity constraints on the facilities can be overcome by using a penalty function as a secondary criterion to evaluate the local optima produced by the solution method (see, for example, Hopkins, 1977). A more direct way of relaxing this assumption but at the cost of added computational effort would be to introduce explicit capacity constraints in (LUDP) or (P) and solve the H resulting transportation problems at step b of QUALF.

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[1] Hopkins (1977) used such a heuristic in his own work.

To summarize, it seems that the only stringent limitation of the present formulation of the land-use-transportation design problem is the one-mode assumption. All the other limitations can be overcome or are not serious limitations.

V Conclusion

Three design problems were discussed in this article. First, it was shown that the network design problem with congestion could be solved with the use of an all-or-nothing traffic assignment algorithm. Computational experience was reported. Secondly the land use design problem was formulated as an extension of the Koopmans-Beckmann problem and a heuristic was proposed to solve that problem and computational experience reported. In both cases the solution algorithms seem promising to solve large, real world problems. And thirdly it was shown that the seemingly more complex problem of designing jointly a land use plan and a transportation network reduces to a pure land use design problem. All that is needed to solve the joint optimization problem is a shortest path code and a heuristic to solve the land use design problem. Therefore the model proposed in this article seems promising to generate alternative sketch plans for land use and transportation. It overcomes some of the limitations of previous models and offers a very simple way of handling computationally the network design problem within the framework of a land use design model. Other land-use design models than the one mentioned in this paper could be used. The only requirement on the land-use model is that in equation (27) one should be able to express demand for travel between the activities located at locations p and q as a simple function of the discrete variables. The central facility location problem could provide such an alternative land-use model, if one assumes that demand activities are already located.

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Appendix

Proof that the BPR function satisfies N4 and N5

N4: $s_a(v_a, z_a) = t_a + t_a\rho \frac{v_a^{\mu}}{z_a^{\mu}}$ with $\mu \geq 1$, $t_a > 0$, $\rho > 0$

$$\frac{\delta^2 s_a}{\delta z_a^2} = \mu(\mu+1) t_a \rho\, v_a{}^{\mu} z_a{}^{-\mu-2} \geq 0$$

for $v_a \geq 0$ and $z_a > 0$

$$\frac{\delta^2 s_a}{\delta v_a^2} = \mu(\mu-1) \frac{t_a\rho}{z_a^{\mu}} v_a{}^{\mu-2} \geq 0$$

for $v_a \geq 0$ and $z_a > 0$

N5: $s_a(v_a, z_a) = t_a + t_a\rho\, u_a^{\mu}$

with $u_a = \frac{v_a}{z_a}$

$$\frac{d\sigma_a}{du_a} = \mu\, t_a\rho\, u_a{}^{\mu-1} > 0 \quad \text{for} \quad u_a > 0$$

$$\frac{d^2\sigma_a}{du_a^2} = \mu(\mu-1)\, t_a\rho\, u_a{}^{\mu-2} \geq 0 \quad \text{for} \quad u_a > 0.$$